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# A Physically Motivated Solution of the Alfvén Problem

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## ABSTRACT

The Alfvén problem has plagued explicit numerical calculations of magnetohydrodynamic fluid behavior in regions where the fluid density is relatively small but can be solved by the inclusion of terms representing the displacement current in Maxwell's Equations. This modification limits the important physical velocities to c in a realistic way and so permits much longer computational timesteps than had previously been possible. Thus the small c technique will greatly extend the usefulness of present day computers for solving these MHD problems.

## PROBLEM STATUS

## AUTHORIZATION

# PHYSICALLY MOTIVATED SOLUTION OF THE ALFVÉN PROBLEM

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The Alfvén problem has plagued explicit numerical calculations of magneto-hydrodynamic fluid behavior in regions where the fluid density is relatively small but can be solved by the inclusion of terms representing the displacement current in Maxwell's Equations. This modification limits the important physical velocities to c in a realistic way and so permits much longer computational time-steps than had previously been possible. Thus the small c technique will greatly extend the usefulness of present day computers for solving these MHD problems.

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For years, the Alfvén problem has seriously limited explicit, multidimensional computations of the magnetohydrodynamic (MHD) behavior of conducting fluids where the mass density $\rho$ or the magnetic field strength $\underline{B}$ varies greatly from one point in the calculational region to another. Problems arise because the Alfvén velocity:

$$\underline{V}_A \equiv \underline{B} / \sqrt{4\pi\rho} \tag{1}$$

can vary widely across a highly inhomogeneous magneto-fluid system requiring a prohibitively small computational timestep, $\delta t$, to provide numerical stability according to the usual stability criterion, [1,2]

$$\delta t < \delta x / \max (V_A(x,t)), \tag{2}$$

where the maximum is taken over the whole system. This stability condition states in general that numerical information about changes in the fluid must propagate faster than the largest of the characteristic fluid velocities. In

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the Alfvén problem under consideration here, $V_A$ is the largest velocity of interest principally because $\rho$ becomes very small in certain regions, say at the edges of a magnetically confined plasma or at the top of a gravitationally stratified atmosphere. A small timestep is not intrinsically disadvantageous, of course, since phenomena happening on the dynamic timescales of the Alfvén waves in the low density regions may be of interest. Usually, however, the physics of primary interest occurs in the high density regions where most of the fluid is located. If the central density is four orders of magnitude greater than the density of the fluid at the wall, as can occur in some plasma-pinch calculations for instance, one may be faced with a timestep which is 100 times smaller than desired simply to provide numerical stability in a region of negligible physical interest anyway.

The stability condition (2) applies to explicit calculations where numerical "information" can propagate only to the nearest-neighbor cells during a single timestep. This defines a numerical velocity

$$V_{num} = \delta x / \delta t.$$

Thus a numerical instability in an explicit computational model is much like a shock. The inability of the computational fluid to propagate energy away from an accumulation point due to a slow numerical speed means that a perturbation in the computed flow can accumulate unendingly until the calculation must be stopped. By increasing the numerical velocity until it is greater than all the characteristic velocities of the fluid, this numerical "steepening" of perturbations can be removed. This usually involves shortening $\delta t$ however.

This Alfvén problem is confronted in all explicit numerical calculations of strongly inhomogeneous magnetofluids when a partial differential equation describing the evolution of the magnetic field must be solved simultaneously with equations describing the fluid motion. Various methods of circumventing the problem have been attempted previously but none has been fully satisfactory.

Using an implicit rather than explicit finite - difference formulation of the relevant equations can have the effect of raising $V_{num}$ to infinity,[3,4] but usually becomes prohibitively complicated in more than one dimension. The programming almost invariably becomes difficult and iterations may be necessary, hence, computational effort increases greatly per timestep(even though the timesteps can be taken over a somewhat longer physical time).

In explicit calculations various artificial corrections have been used but these lack a physical basis and so the properties of the fluid are often grossly altered. For instance, artificial limits on the density set an upper limit to the Alfvén velocity but seriously hinder fluid motions in the artificially dense regions, can cause anomalous instabilities of a convective type, and may damage conservation of density. Another less than ideal fix involves applying an implicit numerical diffusion term to the explicit difference scheme. The propagation speed of wave energy due to the numerics can still be too small for stability, but the implicit diffusion will prevent instability by diffusing away perturbation accumulations before they can become dangerously large. Unfortunately, this has the effect of changing wavelike phenomena to evanescent or diffusion waves wherever the numerical speed becomes too small.

In nature, the Alfvén problem is solved by limiting all important velocities to c, the velocity of light. For the Alfvén and magnetosonic waves of interest here this is done by the displacement current in Maxwell's equations which is usually neglected in MHD models. By including this term, an improved MHD model can be found which does not suffer the Alfvén problem. Consider the following equations:

$$\frac{\partial \rho}{\partial t} = - \underline{\nabla} \cdot (\rho \underline{V}),$$

$$\rho \frac{d\underline{V}}{dt} = - \underline{\nabla} P + \frac{\underline{J} \times \underline{B}}{c}$$

$$\frac{d}{dt} (P/\rho^\gamma) = 0$$

(3)

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$$\frac{1}{c}\frac{\partial B}{\partial t} = -\underline{\nabla} \times \underline{E}$$

$$\frac{\underline{J}}{c} = \frac{1}{4\pi}\underline{\nabla} \times \underline{B} - \frac{1}{4\pi c}\frac{\partial E}{\partial t} .$$

(3) (con't)

Using the definition of the current $\underline{J}$, the continuity equation, and Ohm's law in the simple form $\underline{E} + \frac{\underline{V} \times \underline{B}}{c} = 0$. The momentum and magnetic equations can be rewritten as

$$\frac{\partial(\rho v)}{\partial t} = -\underline{\nabla} \cdot \underline{\underline{\mathbb{P}}} + \frac{1}{4\pi c^2}\frac{\partial}{\partial t}(\underline{V} \times \underline{B}) \times \underline{B}, \tag{4}$$

$$\frac{\partial B}{\partial t} = \underline{\nabla} \times (\underline{V} \times \underline{B})$$

where $\mathbb{P}_{ij} \equiv (\rho T + \frac{B^2}{8\pi})S_{ij} + \rho V_i V_j - \frac{B_i B_j}{4\pi}$

is the usual pressure-stress tensor. The magnetic equation is unchanged. The last term in the momentum equation (4a) arises from the displacement current. This term may be rearranged using the magnetic equation to give a modified momentum equation,

$$\frac{\partial}{\partial t}\left[\rho\underline{V} - \frac{(\underline{V} \times \underline{B}) \times \underline{B}}{4\pi c^2}\right] = -\underline{\nabla} \cdot \underline{\underline{\mathbb{P}}} - \underline{\nabla} \cdot \underline{\underline{\mathbb{P}}}_D$$

where

$$\mathbb{P}_{Dij} \equiv \frac{1}{4\pi c^2}\left[(\underline{V} \times \underline{B})_i(\underline{V} \times \underline{B})_j - \frac{|\underline{V} \times B|^2}{2}\delta_{ij}\right]$$

(5)

is the displacement-current correction to the stress tensor.

The physical importance of Eq. (5) arises from the inclusion of the electromagnetic effects in conservative form. The time rate of change of a generalized momentum is equal to the divergence of a generalized pressure-stress tensor. While the correction $\underline{\underline{\mathbb{P}}}_D$ is essentially relativistic, being of order $(V/c)^2$ relative to other stress tensor terms, the Poynting vector contribution to the generalized momentum is basically nonrelativistic, depending only on large fields or low density to be important. In fact, if the electromagnetic field is thought of as having mass, one can write a generalized mass-density tensor $\underline{\underline{\rho}}^*$ from Eq. (5),

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$$\rho^*_{ij} = \rho\,\delta_{ij} + \frac{B^2}{4\pi c^2}\,\delta_{ij} - \frac{B_i B_j}{4\pi c^2} \quad . \tag{6}$$

The generalized momentum is then $\underline{\underline{\rho}}^* \cdot \underline{v}$ .

The computational importance of Eq. (5) can be seen best by solving for the linear dispersion relations of the Alfvén and magnetosonic model which are found from the following linear equation[5]

$$-\omega^2 \underline{v} + \omega^2\,(\underline{v} \times \frac{V_A}{c}) \times \frac{V_A}{c} + (S^2 + V_A{}^2)(\underline{k}\cdot\underline{v})\underline{k}$$

$$+ (\underline{k}\cdot\underline{V}_A)\left[(\underline{k}\cdot\underline{V}_A)\underline{v} - (\underline{V}_A\cdot\underline{v})\underline{k} - (\underline{k}\cdot\underline{v})\underline{V}_A\right] = 0. \tag{7}$$

Here $S^2 \equiv \dfrac{\gamma P_0}{\rho_0}$ is the square of the sound speed and $\underline{V}_A$ (Eq. (1)) is the vector Alfvén velocity. Assuming that $\underline{k}$ is perpendicular to both $\underline{v}$ and $\underline{B}$ gives the dispersion relation for transverse Alfvén waves,

$$\left(\frac{\omega}{k}\right)^2_{TA} = \gamma V_A^2\,\cos^2\theta \tag{8}$$

where $\gamma \equiv (1 + V_A^2 / c^2)^{-1}$ and where $\theta$ is the angle between $\underline{k}$ and $\underline{B}$. As $\rho$ approaches zero, $\gamma$ approaches $c^2/V_A^2$ and

$$\left(\frac{\omega}{k}\right)^2_{TA} \to c^2\cos^2\theta, \tag{8a}$$

a finite rather than infinite phase velocity.

The four magnetosonic (magnetoacoustic) roots are found under the assumption $\underline{v}$ lies in the plane established by $\underline{k}$ and $\underline{B}$. After some algebra

$$\left(\frac{\omega}{k}\right)^2_{MS} = (S^2\cos^2\theta + \gamma S^2\sin^2\theta + \gamma V_A^2)/2$$

$$\pm \frac{1}{2}\sqrt{(S^2\cos^2 + \gamma S^2\sin^2\theta + \gamma V_A^2)^2 - 4\gamma\cos^2\theta S^2 V_A^2}. \tag{9}$$

This reduces to the usual result[5,6] when $\gamma = 1$ and, when $\gamma$ becomes small $(V_A/c \gg 1)$,

$$\left(\frac{\omega}{k}\right)^2_{FMS} \cong c^2,$$

$$\left(\frac{\omega}{k}\right)^2_{SMS} \cong \frac{c^2 S^2 \cos^2\theta}{c^2 + S^2 \cos^2\theta} \tag{10}$$

where F and S refer to "Fast" and "Slow" magnetosonic modes respectively. Notice that the slow mode is primarily compressional parallel to $\underline{B}$ and does not propagate across the magnetic field. As for the Alfvén modes, all velocities are less than or equal to c.

Various levels of numerical approximation are possible. The Alfvén problem will be solved, of course, if Eqs. (4), (5) and (6) are used as they stand, but the relativistic correction $\underline{\underline{\mathbb{P}}}_D$ is not strictly necessary unless $(V/c)^2 \sim 1$ in which case the definition of fluid momentum must be slightly modified. Thus $\underline{\underline{\mathbb{P}}}_D$ can usually be dropped to greatly simplify computation. Another simplification can be made which further reduces calculation. The off diagonal term $B_i B_j$ in $\underline{\underline{z}}^*$ can be set to zero. This reduces calculation greatly but gives linearized dispersion relations which are not strictly correct. The slow magnetosonic modes are primarily acoustic (compressional) when $V_A \gg S$. Since the use of a scalar $\rho^*$ slows down all modes equally in low density regions, these compressional modes are slowed but really should not be. This effect, although spurious, is not likely to be very important in

(1) Low density regions where an artificially small value of c may be required to allow sufficiently long timesteps and where the physics is often of little interest anyway;

(2) High density regions where $V_A \sim S$, where c is much larger than either, and thus the difference of $\rho^*$ from $\rho$ is negligible anyway.

The idea of using an artificially small value of c is well established, being rather like the use of mass ratio 100 for manybody plasma simulations involving the individual trajectories of many ions and electrons. In the present

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case all of the important physical effects are preserved and can be scaled analytically to the correct value of c in those problems where an artificial value is used for c.

Figure 1 shows the numerical Alfvén-wave phase velocity (x's) computed using the scalar $\rho*$ correction for a simple plane-wave Alfvén excitation.[7] The phase velocities relative to c are plotted versus $V_A/c$; the solid curve is the theoretical dispersion relation. The asymptote $V_\phi = V_A$ is also shown. In the configuration ($B_x$ = constant, $\rho$ = constant, $V_y$ perturbation sinusoidal in X) the dispersion relations are as given earlier for the transverse Alfvén and fast magnetosonic modes. The slow magnetosonic modes will of course be slowed artifically since the off diagonal $\underline{\rho}*$ terms have been neglected.

Figure 2 shows the perturbation amplitude as a function of time. The slow decay of the wave shows the effect of a small stabilizing viscosity. The frequencies and hence phase velocities shown in Fig. 1 were taken from a series of graphs similar to Fig. 2. The systematic difference between the numerical points and theoretical curve in Fig. 1 is of order 3% and is easily explained by the finite difference approximations employed for the temporal and spatial sinusoids.

A physical solution to the Alfvén problem in explicit, multidimensional computer calculations of low density MHD fluids has been proposed and tested. The inclusion of terms arising from the physical displacement current in Maxwell's equations limits physically important velocities to c, the velocity of light, and therefore much larger computational timesteps can be used than was heretofore possible. Factors of 10 have been realized in calculations where $\rho_{max}/\rho_{min} \sim 10^4$.

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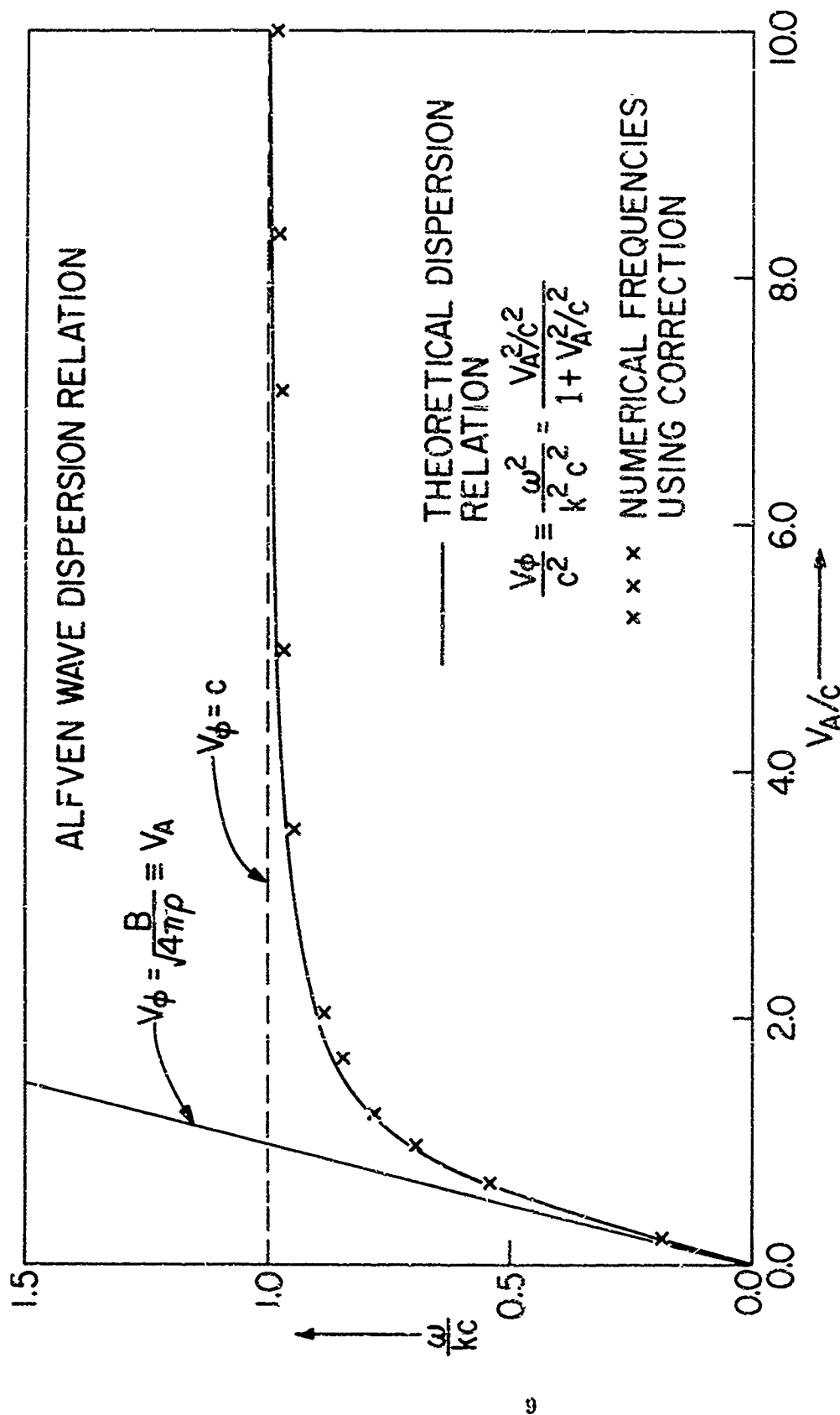
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Fig. 1 - Computational and theoretical Alfven-wave dispersion relations. The systematic error between theory (solid line) and computation (X's) is accounted for by the finite-difference representation of trigonometric functions.
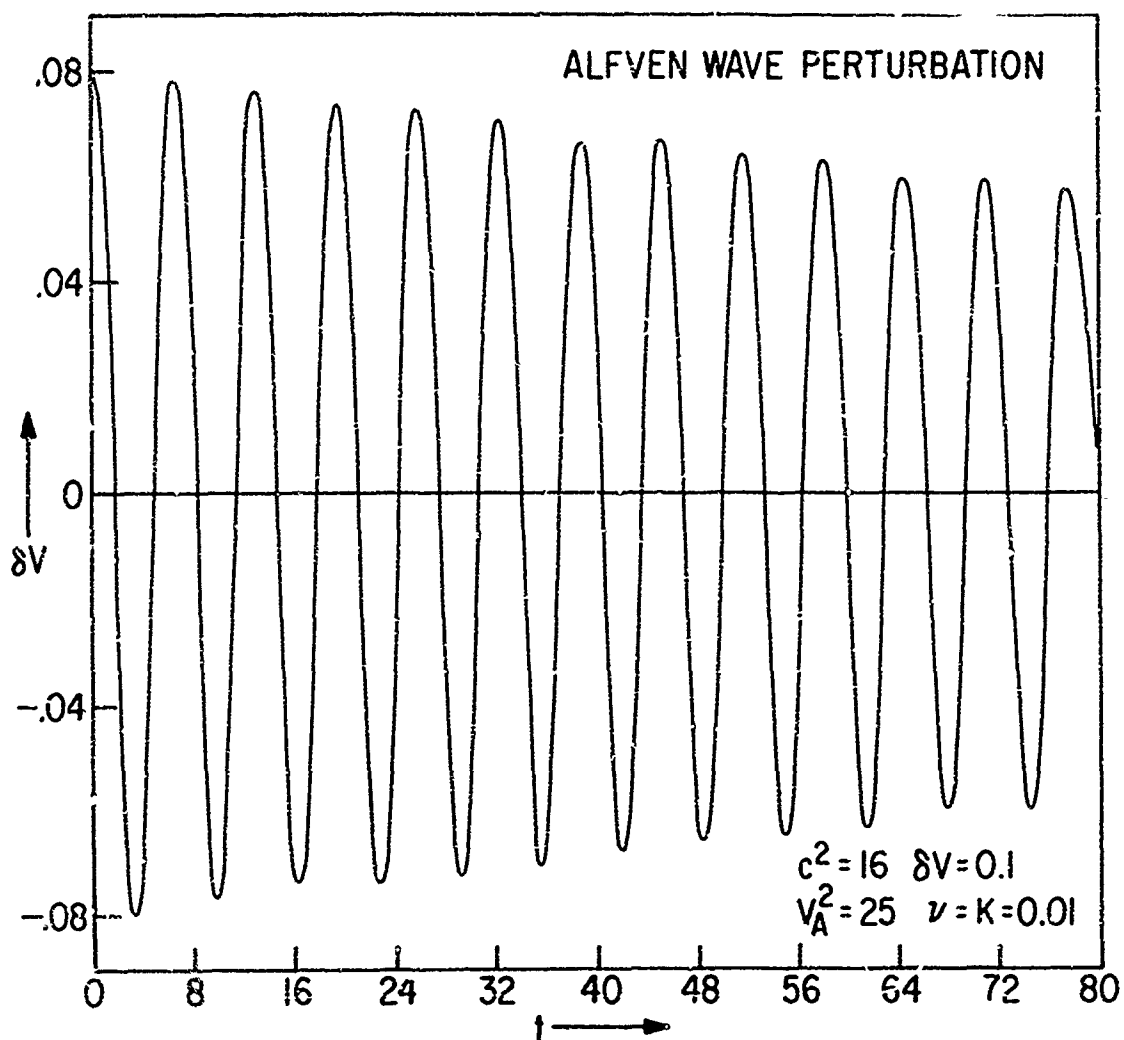
Fig. 2 – Computational Alfven-wave amplitude as a function of time. Non-zero viscosity and thermal conductivity account for the slow decay of the oscillation.

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